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Existence of naked singularities in spherical symmetry

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Abstract

Existence conditions for naked singularities in spherical symmetry will be proved for arbitrary Misner–Sharp masses which fulfil the weak energy condition.

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1. Introduction

The final state of gravitational collapse is an important open issue of classical gravity. It is, in fact, commonly believed that a collapsing star that is unable to radiate away—via, e.g. supernova explosion—a sufficient amount of mass to fall below the neutron star limit, will certainly and inevitably form a black hole such that the singularity corresponding to diverging values of energy and stresses will be safely hidden—at least to far away observers—by an event horizon. However, this is nothing more than a conjecture—what Roger Penrose first called a “Cosmic censorship” conjecture [3]—and has never been

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proved. On the contrary, in the last 20 years of research, many analytic examples of spherically symmetric naked singularities satisfying the principles of physical reasonableness have been discovered (cf., e.g. [2] and references therein).

So the first results for a collapse of spherical dust clouds were given by Eardley and Smarr [8] using numerical techniques, and hitherto in an analytic way by Christodoulou [9] preparing the tools to characterize completely the spectrum of the endstates. However, dust models do lack their physical reliability, since it is supposed that anisotropic stresses occur during the collapse, and for the few results regarding perfect fluids on this field. So it was in [5,6] that only tangential stresses were considered.

A complete, new model of gravitational collapse which includes both radial and tangential stresses is done in [1]. The authors derive a class of anisotropic solutions which is in itself new, and contains the dust and the tangential stress metrics as special cases. The study of gravitational collapse is done under a non-degenerate condition of the energy profile of the initial data. In this paper we develop the study, started in [1], considering the general case under the degenerate condition.

We proof conditions for all masses satisfying standard requirements as the weak energy condition and regularity in the center in terms of the first non-vanishing Taylor term proving the existence or non-existence of local super-solution for Eq. (14). Throughout all sections we assume that the non-degenerate condition does not hold, that is we assume $\alpha = 0$ (cf. (13)). However, for the discussion of the endstates in the proof of Theorem 9 (the main result of the present paper) the general case (including the non-degenerate case) can be deduced from it.

2. Einstein-field equations in spherical symmetry

In the following section we sketch out some notations and facts from Giambo et al. [1]. Any spherically symmetric hyperbolic metric in comoving coordinates takes the form

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu}\,\mathrm{d}t^2 + \left(\frac{1}{\eta}\right)\mathrm{d}r^2 + R^2(\mathrm{d}\theta^2 + \sin\theta^2\,\mathrm{d}\varphi^2) \tag{1}$$

with functions ν, η and R depending on r, t, the comoving radius and time, respectively. In this frame the physical properties of elastic materials can be described in terms of a state function that in the comoving gauge of spherical symmetry reads as function $w(r, R, \eta)$ depending only on the radius and two strain parameters [4,5]. In this gauge the equations of state become

$$p_r = 2\rho\eta\frac{\partial w}{\partial \eta}, \qquad p_\varphi = -\frac{1}{2}\rho R\frac{\partial w}{\partial R}. \tag{2}$$

As was shown in [1] another gauge proofs very useful dealing with gravitational collapses. In the area-radius gauge—first introduced by Ori [7]—the comoving time t is substituted by R such that the line element (1) becomes

$$\mathrm{d}s^2 = -A\,\mathrm{d}r^2 - 2B\,\mathrm{d}R\,\mathrm{d}r - C\,\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin\theta^2\,\mathrm{d}\varphi^2) \tag{3}$$

with functions A, B and C depending on r and R. The choice of the gauge and the fundamental field variables is strongly related to the field equations in the way the latter ones can be simplified. In fact, introducing the velocity of the transformed field $v^\mu = \mathrm{e}^{-\nu}\dot{R}\delta^\mu_t$ as its modulo $u = |\mathrm{e}^{-\nu}\dot{R}|$ and the area-radius slice $\Delta = B^2 - AC = 1/\eta u^2$, and choosing a convenient set of the field equations, they can be expressed in terms of A, u and $\sqrt{\Delta}$. A useful quantity is the Misner–Sharp mass defined by $\Psi = (R/2)(1 - A/\Delta)$ that characterize implicitly the apparent horizon during the collapse. Moreover, it is by Ψ that the field variables decouple, and by a separation ansatz for the state function w, the system can be integrated to the admissible class of metrics

$$\mathrm{d}s^2 = -\left(1 - \frac{2\Psi}{R}\right)\Delta\,\mathrm{d}r^2 + \sqrt{\Delta}\frac{Y}{u}(\mathrm{d}R\,\mathrm{d}r + \mathrm{d}r\,\mathrm{d}R) - \frac{1}{u^2}\,\mathrm{d}R^2 + R^2\,\mathrm{d}\Omega^2, \tag{4}$$

where

$$\sqrt{\Delta}(r, R) = \int_r^R \frac{\partial(1/u)}{\partial r} Y(r,\sigma)^{-1}\,\mathrm{d}\sigma + \frac{1}{Y(r,r)u(r,r)}, \tag{5}$$

and

$$Y^2 = 1 + u^2 - \frac{2\Psi}{R}. \tag{6}$$

Both, Y as well as Ψ are independent arbitrary functions. In order to render the model physically reasonable the weak energy condition must be satisfied. To this end it is useful to express the state function in terms of the Misner–Sharp mass and of the distribution of the initial mass E

$$w = E\left(\frac{\Psi_{,r}}{Y} + \frac{\Psi_{,R}}{\sqrt{\eta}}\right). \tag{7}$$

From the pressure and the energy density $\epsilon = \rho w$ (given by the state function and the internal mass density ρ) the weak energy condition in spherical symmetry is equivalent to $\epsilon \geq 0, \epsilon + p_r \geq 0, \epsilon + p_\varphi$, leading to

$$\Psi_{,r} \geq 0, \qquad \Psi_{,R} \geq 0, \qquad \Psi_{,r} \geq \frac{R}{2}Y\left(\frac{\Psi_{,r}}{Y}\right)_{,R}, \qquad \Psi_{,R} \geq \frac{R}{2}\Psi_{,RR}. \tag{8}$$

Further, the center is supposed to be regular for all times up to the formed singularity and the stress tensor has to be isotropic at $r = 0$. By this and together with the condition that the initial energy decreases with increasing radius one states easily the following set of conditions that defines together with the state of matter equation the *area-radius separable space–time* (*ARS*)

$$Y(0,0) = 1, \qquad \Psi(0,0) = 0, \qquad \mathrm{D}\Psi(0,0) = 0, \qquad \mathrm{D}^2\Psi(0,0) = 0, \tag{9}$$

$$\Psi_{,rr}(r,r) + 2\Psi_{,rR}(r,r) + \Psi_{,RR}(r,r) - \frac{2}{r}(\Psi_{,r}(r,r) + \Psi_{,R}(r,r)) \leq 0. \tag{10}$$

2.1. Conditions for singularity forming

We recall that a singularity forms if the energy density diverges which is given if R or $u\sqrt{\Delta}$ vanish. The latter case leads to $R'(r,t) = 0$, and singularities of this kind are called *shell-crossing* singularities. In this paper we concentrate us on the former case $R = 0$, that is on the so-called *shell-focusing* singularities. Thus, it is sufficient to require strict positivity of the $\sqrt{\Delta(r,0)}$ together with a non-increasing behavior of $\sqrt{\Delta(r,R)}$ w.r.t. R. Furthermore, the implicit time curve $t_s(r)$ for the zeros of R must be finite in $r = 0$ if any singularity should occur. This, by integrating along the flow lines, leads to the condition

$$0 < \lim_{r\to 0^+} \int_0^r e^{-\nu(r,\sigma)} \mathcal{H}(r,\sigma)\, d\sigma, \quad \mathcal{H} := \frac{1}{u}. \tag{11}$$

Since time reparametrizations are always possible such that holds $\nu(0,t) = 0$. Moreover, by the equation

$$Y\nu' = R'Y_{,R}$$

(stemming from the field equation $\dot{R} = \dot{R}\nu' + R'\dot{\lambda}$), we deduce that ν is uniformly bounded and $\lim_{r\to 0^+} \nu(r,\sigma) = 0$ uniformly for $\sigma \in [0,r]$. Therefore we obtain the equivalent condition

$$\lim_{r\to 0} \int_0^r \mathcal{H}(r,\sigma)\, d\sigma < \infty. \tag{12}$$

As shown in [1], for an ARS space–time this is fulfilled if the non-degenerate condition

$$DY(0,0) = 0, \qquad \alpha := \frac{\partial^3}{\partial r^3}\Psi(0,0) > 0 \tag{13}$$

is satisfied. As will be shown below, the case $\alpha > 0$ can be dropped and condition (12) is true even in the degenerate case $\alpha = 0$. We prove equivalent conditions for this case in Lemma 3.

For the sake of a consistent terminology we refer to [1] and call a ARS space–time *collapsing* under the conditions that shell-crossing does not occur and if holds (12).

Finally we consider the radial null-geodesic equation

$$\frac{\partial R}{\partial r} = u\sqrt{\Delta}(Y - u). \tag{14}$$

We recall the definition of nakedness from Giambo et al. [1] and say that

Definition 1. The center is called (locally) naked if there exists a future pointing local solution R_g of (14) with $R_g(0) = 0$ and $R_h(r) < R_g(r)$ for $r > 0$, where R_h is the implicit function for $R = 2\Psi$.

In order to answer the question whether a formed singularity will be hidden or visible to nearby observers it is now sufficient to state existence (or non-existence) of sub- and super-solutions for (14) for sufficiently small radii r. We recall that for a sub-solution R_+ (or super-solution R_-, resp.) for (14) with initial value $R_\pm(0) = 0$ holds $\partial R_+/\partial r \le u\sqrt{\Delta}(Y - u)$

(and $\geq$ resp.). In fact, the ill-posedness of the Cauchy problem in the center, where the singularity forms, does not assure existence results for geodesics. Nevertheless, local radial null-geodesics (if there are any) away from the center must be bounded and "traced back" to the formed singularity in the center by some sub- and super-solutions (see Lemma 8 and the main Theorem 9, and Lemma 2 in [1] for details).

Hence, due to the local existence argument it is sufficient to extend the right-hand side in (14) in Taylor and to establish conditions on the first order terms. As done in [1] the spectrum of the endstates is fully characterized under the non-degenerate condition $\alpha > 0$

Theorem 2 ([1]). *In a collapsing space–time the singularity forming in the center is locally naked if $n < 3$ or if $n = 3$ and $\xi > \alpha\xi_c$, where $\sqrt{\Delta}(r, 0) = \xi r^{n-1} + O(r^k)$, $k \geq n$, and $\xi_c = (26 + 15\sqrt{3})/2$. If $n > 3$, then the singularity is covered.*

Nevertheless, dropping the non-degenerate condition we prove conditions to obtain a similar result as in Theorem 9, the main result of the present paper. As will come out from the discussion in the proof there one easily deduces also the non-degenerate case (see Remark 10 after the proof of Theorem 9).

3. Expansion of $\sqrt{\Delta}$

In this section the Taylor expansion for $\sqrt{\Delta}$ will be derived for ARS space–times under the condition that in (13) holds $\alpha = 0$. As in (5) we have

$$\sqrt{\Delta}(r, 0) = \int_0^r \frac{\mathcal{H}_{,r}(r, \sigma)}{Y(r, \sigma)}\, \mathrm{d}\tau + \frac{\mathcal{H}(r, r)}{Y(r, r)},$$

where by the definition in (11) and (6)

$$\mathcal{H}(r, \sigma) = \sqrt{\frac{\sigma}{2\Psi + \sigma(Y^2 - 1)}},$$

and for Y near 0 holds $Y(r, \sigma) = 1 + O(r^2) + O(r\sigma) + O(\sigma^2)$. For simplicity we define $H(r, \sigma) := 2\Psi(r, \sigma) + \sigma(Y(r, \sigma)^2 - 1)$ and derive the Taylor like expansion

$$H(r, R) = \sum_{i+j\geq 3} T_{ij} r^i R^j + R \sum_{i+j\geq 2} \Lambda_{ij} r^i R^j \tag{15}$$

where the coefficients correspond to the expansion terms of 2Ψ and Y^2, respectively. The relation between the coefficients for Y and Y^2 is clearly given by

$$\Lambda_{i,k-i} = \sum_{l=0}^{k} \sum_{\substack{r+s=i \\ 0\leq r\leq l \\ 0\leq s\leq k-l}} \tilde{\Lambda}_{r,l-r} \tilde{\Lambda}_{s,(k-l)-s}, \tag{16}$$

where $\tilde{\Lambda}$ denote the coefficients for Y.

Now consider the implicit time function, for which the shell of label r vanishes in the center. Using $\nu(r,\sigma) = \mathrm{O}(r)$ for every σ near 0 we have

$$t_s(r) = \int_0^r \mathrm{e}^{-\nu(r,\sigma)}\mathcal{H}(r,\sigma)\,\mathrm{d}\sigma \simeq \int_0^r \mathcal{H}(r,\sigma)\,\mathrm{d}\sigma. \tag{17}$$

In order that the singularity in the center forms in finite time it must hold

$$t_* := \lim_{r\to 0} t_s(r) \in (0,\infty).$$

For notational convenience we set $\beta := T_{21} + \Lambda_{20}$, $\gamma := T_{12} + \Lambda_{11}$ and $\delta := T_{03} + \Lambda_{02}$ and define

$$l := \min\{i \in \mathbb{N} | T_{i0} \neq 0\} \geq 4. \tag{18}$$

We prove the following lemma.

Lemma 3. *Suppose* $\alpha = 0$. *Then* $t_* \in (0,\infty)$ *if and only if holds*: $\beta > 0$, *or if* $\beta = 0$, *then* $\gamma > 0$.

Proof. Applying the transformation $\sigma = r\tau$, $\tau \in (0, 1)$ we first get

$$H(r, r\tau) = \sum_{k\geq 3} r^k \sum_{i+j=k} T_{ij}\tau^j + \sum_{k\geq 2} r^{k+1} \sum_{i+j=k} \Lambda_{ij}\tau^{j+1} = r^l T_{l0} + \Theta(\tau)r^3 + \mathrm{O}(r^{s+3}), \tag{19}$$

where $s \geq 1$ and

$$\Theta(\tau) := \beta\tau + \gamma\tau^2 + \delta\tau^3$$

and the integrand on the right-hand side of (17) reads as

$$\mathcal{H}(r,\sigma)\,\mathrm{d}\sigma = \sqrt{\frac{\tau}{\Theta(\tau) + Q(r)}}\,\mathrm{d}\tau \tag{20}$$

with error

$$Q(r) := P(\tau)r^s + T_{l0}r^{l-3} := \sum_{\substack{k\geq s\\ i+j=k}} h_{ij}\tau^j r^s + T_{l0}r^{l-3},$$

$$Q(r) = \mathrm{O}(r^p), \qquad p := \min\{l-3, s\}, \tag{21}$$

where

$$h_{ij} = \begin{cases} T_{ij} + \Lambda_{i,j-1}, & j \geq 1, \\ T_{i0}, & j = 0. \end{cases}$$

Clearly for any sequence $r_n \to 0$ the integral exists if and only if the polynomial Θ has non-negative first order coefficient β. In particular it has positive limit if and only if holds strict positivity, or if it is zero, then must hold $\gamma > 0$. □

Next we prove conditions for $\sqrt{\Delta}$, in order to hold some expansion in 0.

Lemma 4. *Suppose* $\alpha = 0$, $\beta > 0$. *Let* $s \geq 1$, l, p *and* Q, P *be as defined in* (18), (19) *and* (21), *respectively, and let* $Q \geq 0$. *Then there exists an integer* $q \geq 2$ *and* $\xi \in \mathbb{R}$ *such that under the conditions*

(i) $l > p/2 + 4, q > p/2, s > 1, s - 1 > p/2,$

(ii) $\begin{cases} P(0) \geq 0 & if\ l - 3 < s, \\ (P(0) - T_{l0} \neq 0,\ P(0) > 0) \text{ or } P(0) = 0 & if\ s = l - 3, \\ P(0) > 0 & if\ s < l - 3, \end{cases}$

the following estimate holds:

$$\xi r^{n-1} + \mathrm{O}(r^k) \leq \sqrt{\Delta}(r, 0) \leq \mathrm{O}(r^\nu), \quad k > n - 1,\ \nu > 0 \tag{22}$$

Proof. Split $\sqrt{\Delta}(r, 0)$ up into

$$\sqrt{\Delta}(r, 0) = \int_0^r g(r, \sigma)\mathcal{H}_{,r}(r, \sigma)\,\mathrm{d}\sigma + \frac{1}{Y(r, r)}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \mathcal{H}(r, \sigma)\,\mathrm{d}\sigma, \tag{23}$$

$$g(r, \sigma) := \frac{1}{Y(r, \sigma)} - \frac{1}{Y(r, r)}. \tag{24}$$

By the transformation $\sigma = r\tau$ and by the definition of $\mathcal{H}$ we get

$$\frac{\partial \mathcal{H}}{\partial r}(r, \tau r) r\,\mathrm{d}\tau = -\frac{r\sqrt{r\tau}}{2}\frac{(\partial H/\partial r)(r, r\tau)}{H^{3/2}(r, r\tau)}\,\mathrm{d}\tau.$$

It holds

$$\frac{\partial H}{\partial r}(r, r\tau) = lr^{l-1}T_{l0} + (2\beta\tau + \gamma\tau^2)r^2 + M(r),$$

where

$$M(r) := \sum_{\substack{k=i+j\geq s+2 \\ i\geq 1}} i\,h_{ij}\,\tau^j r^k = \mathrm{O}(r^{s+2}).$$

For $l \geq 4$ we have

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial r} r &= -\frac{r\sqrt{r\tau}}{2}\frac{lT_{l0}r^{l-1} + (2\beta\tau + \gamma\tau^2)r^2 + M(r)}{(r^l T_{l0} + \Theta r^3 + \mathrm{O}(r^{s+3}))^{3/2}} \\ &= -\frac{\sqrt{\tau}}{2(Q(r) + \Theta(\tau))^{3/2}}(lT_{l0}r^{l-4} + r^{-1}(2\beta\tau + \gamma\tau^2) + r^{-3}M(r)). \end{aligned} \tag{25}$$

Now

$$g(r, r\tau) \simeq Y(r, r) - Y(r, \tau r) = \sum_{k\geq 2} S_k(\tau) r^k,$$

where $S_k(\tau) = \sum_{i+j=k,\, j>0} \tilde{\Lambda}_{ij}(1 - \tau^j)$. Let be

$$q := \min\{k \geq 2 | S_k \not\equiv 0\}.$$

(Observe that q does not depend on coefficients of the type $\tilde{\Lambda}_{i0}$.) We get

$$\int_0^1 g(r, r\tau)\mathcal{H}_{,r}(r, r\tau) r \,\mathrm{d}\tau = \int_0^1 S_q(\tau)\mathcal{H}_{,r}(r, r\tau) r^{q+1} \,\mathrm{d}\tau + \mathrm{O}(r^{q+1}) \int_0^1 S_{q+1}(\tau)\mathcal{H}_{,r}(r, r\tau) r \,\mathrm{d}\tau. \quad (26)$$

Thus by using (25)

$$\begin{aligned}\int_0^1 S_q(\tau)\mathcal{H}_{,r}(r, r\tau) r^{q+1} \,\mathrm{d}\tau \simeq & - r^{q+l-4} \int_0^1 \frac{l\, T_{l0}\, S_q(\tau)\sqrt{\tau}\,\mathrm{d}\tau}{2(\Theta(\tau) + Q(r))^{3/2}} & =: F(r) \\ & + r^{q-1} \int_0^1 \frac{(2\beta\tau + \gamma\tau^2)\, S_q(\tau)\sqrt{\tau}\mathrm{d}\tau}{2(\Theta(\tau) + Q(r))^{3/2}} & =: G(r) \quad (27) \\ & + r^q \int_0^1 \frac{r^{-3} M(r, \tau) S_q(\tau)\sqrt{\tau}\,\mathrm{d}\tau}{2(\Theta(\tau) + Q(r))^{3/2}} & =: H(r)\end{aligned}$$

The integrals exist, since $Q > 0$ for sufficiently small $r > 0$. The second term in (26) has only higher order terms with respect to the lowest order term in (27).

The second integral $G(r)$ gives a contribute of order $\mathrm{O}(r^{q-1})$, i.e.

$$G(r) \simeq r^{q-1} a, \quad a := \int_0^1 \frac{(2\beta\tau + \gamma\tau^2) S_q(\tau)\sqrt{\tau}\,\mathrm{d}\tau}{2\Theta(\tau)^{3/2}}. \quad (28)$$

For the first integral $F(r)$ we obtain, omitting constants

$$r^{q+l-4} \int_0^1 \frac{\sqrt{\tau}\,\mathrm{d}\tau}{(\Theta(\tau) + Q(r))^{3/2}} \leq \int_0^1 \frac{\sqrt{\tau}}{\Theta(\tau)} \frac{r^{q+l-4}}{\sqrt{P(\tau) r^s + T_{l0} r^{l-3}}} \,\mathrm{d}\tau = \mathrm{O}(r^{q+l-4-p/2}) \quad (29)$$

and hence by conditions (i) and (ii) convergence to 0. By the same arguments it follows that $H(r) \leq \mathrm{O}(r^{q-p/2})$ and again by (i) convergence to 0. On the other hand, there exists $C \in \mathbb{R}$

$$F(r) \geq r^{q+l-4} \int_0^1 \frac{\sqrt{\tau}\,\mathrm{d}\tau}{(C + Q(r))^{3/2}} \simeq \mathrm{O}(r^{q+l-4}) \quad (30)$$

and analogous $H(r) \geq \mathrm{O}(r^{q-1})$.

The second part of the split in (23) reads as

$$\frac{1}{Y(r,r)}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^1 \mathcal{H}(r,r\tau)r\,\mathrm{d}\tau \simeq \frac{\mathrm{d}}{\mathrm{d}r}\int_0^1 \mathcal{H}(r,r\tau)r\,\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}r}\int_0^1 \sqrt{\frac{\tau}{\Theta(\tau)+Q(r)}}\,\mathrm{d}\tau$$
$$= -\frac{1}{2}\int_0^1 \frac{(\mathrm{d}/\mathrm{d}r)Q(r)\sqrt{\tau}\,\mathrm{d}\tau}{(\Theta(\tau)+Q(r))^{3/2}}.$$

The last integral reads as

$$-\frac{1}{2}\int_0^1 \frac{T_{l0}r^{l-4}\sqrt{\tau}\,\mathrm{d}\tau}{(\Theta(\tau)+Q(r))^{3/2}} - \frac{1}{2}\int_0^1 \frac{P(\tau)sr^{s-1}\sqrt{\tau}\,\mathrm{d}\tau}{(\Theta(\tau)+Q(r))^{3/2}} =: I(r)+J(r),$$

where we denote first term by $I(r)$ and the second term by $J(r)$, respectively. Clearly they exist for all $r>0$ and we obtain for I with analogous estimates as in (29) and (30)

$$\mathrm{O}(r^{l-4}) \le I(r) \le \mathrm{O}(r^{l-4-p/2})$$

with convergence to 0 by (i).

For J we obtain, if $P(0)=0$

$$J(r) \simeq sr^{s-1}b, \quad b := -\frac{1}{2}\int_0^1 \frac{P(\tau)\sqrt{\tau}\,\mathrm{d}\tau}{\Theta^{3/2}}.$$

and, if $P(0)>0$

$$J(r) \simeq sr^{s-1-p/2}b, \quad b := \lim_{r\to 0} -\frac{1}{2}\int_0^1 \frac{\sqrt{\tau}}{\Theta}\frac{P(\tau)\,\mathrm{d}\tau}{\sqrt{Qr^{-p}}}$$

and by (i) in both cases convergence to 0.

Finally, we get for $r>0$ small enough by (29) and the above discussion

$$\mathrm{O}(r^{\sigma}) \le F(r)+H(r)+I(r) = \mathrm{O}(r^{\nu})$$

for some $0<\nu\le\min\{l-4-p/2, q-p/2\}$ and $0<\sigma\le\min\{q-1, l-4\}$, and therefore by (28) and by the estimates on J in the case $P(0)=0$ (case $P(0)>0$ is treated similarly)

$$\sqrt{\Delta(r,0)} \simeq F+G+H+I+J \ge \xi r^{n-1} + \mathrm{O}(r^{n-1+\epsilon}),$$
$$\begin{cases} \le ar^{q-1} + sr^{s-1}b + \mathrm{O}(r^{\nu}), \\ \ge ar^{q-1} + sr^{s-1}b + \mathrm{O}(r^{\sigma}) \end{cases}$$

for some $\epsilon \ge 0$, $\xi \ne 0$ and $n := \min\{q, s, \sigma+1\}$. Whence, estimates (22) hold. □

Remark 5. It is easy to check that from the conditions (i) and (ii) in Lemma 4 follows that:

$$l > 5 \quad \text{and} \quad s > 2, \tag{31}$$

and otherwise no estimates for $\sqrt{\Delta}$ of the type (22) can be expected. Therefore we call, $l \in \mathbb{N}$ admissible if it satisfies the conditions in Lemma 4. Furthermore, from condition $Q \geq 0$ follows $P, T_{l0} \geq 0$. Hence, we consider in the following only those masses Ψ for which $T_{l0} \geq 0$.

Remark 6. From $q > p/2$ one gets

$$\tilde{\Lambda}_{ij} = 0 \quad \forall i + j \leq \tfrac{1}{2}p, \ \ j > 0.$$

By (16) this means

$$\Lambda_{i,k-i} = \begin{cases} \sum_{l=0}^{k} \tilde{\Lambda}_{l0}\tilde{\Lambda}_{k-l,0}, & i = k, \\ 0, & i < k \end{cases}$$

for $k \leq p/2$.

4. The horizon and naked singularities

In this section it will be shown that the implicit function for the horizon R_h is a super-solution of the null-geodesic equation near the singularity and that there are sub-solutions of the form cr^l with l defined in (18).

Lemma 7. *The implicit horizon function* R_h *has the same order in r as* $\Psi(r, 0)$. *More precisely holds*

$$R_\mathrm{h}(r) = T_{l0}r^l + \mathrm{O}(r^{l+1}), \quad \omega \in \mathbb{R}_+. \tag{32}$$

thus R_h *is a super-solution for the null-geodesic equation*.

Proof. We have by simple Taylor expansion and using the definition of R_h

$$R_\mathrm{h}(r) = 2\Psi(r, 0) + 2\frac{\partial\Psi(r, 0)}{\partial R}R_\mathrm{h}(r) + \mathrm{O}(|R_\mathrm{h}(r)|^2), \tag{33}$$

where

$$\Psi(r, 0) = \mathrm{O}(r^l), \qquad \frac{\partial\Psi(r, 0)}{\partial R} = \mathrm{O}(r^d), \quad d \geq 2.$$

By the weak energy condition the horizon R_h vanishes only in 0. Divide (33) by $R_\mathrm{h}(r)$. This proofs (32). From (6) follows that $Y = u$, hence the horizon is a super-solution. □

In the following we set for convenience $\tilde{\alpha} := T_{l0} \geq 0$ (recall Remark 5).

Lemma 8. *The central singularity of an ARS space–time is naked if and only if there exists a sub-solution for* (14) *of the form* cr^l *such that* $c > \tilde{\alpha}$.

Proof. The proof can be readily seen in [1] as a slightly modified version of Lemma 2. □

Theorem 9. *Let* $\tilde{\alpha}$ *as above and*

$$\xi r^{n-1} + \mathrm{O}(r^k) \leq \sqrt{\Delta}(r, 0) \leq \mathrm{O}(r^\nu), \quad k > n-1,\ \nu > 0,\ \xi \in \mathbb{R}. \tag{34}$$

and $l := \min\{i \geq 0 : T_{i0} \neq 0\}$ *an admissible number* (*see* Remark 5). *In a collapsing ARS space–time, the singularity forming at the center is locally naked if,* $1 \leq n-1 \leq l$. *If,* $l = n$, *then there exists some positive number* $\mu = \mu(l) \in \mathbb{R}$ *such that if* $\xi > 0$ *and if holds* $\xi/\tilde{\alpha} > 1/\mu$, *then the singularity is naked.*

Proof. We proof that cr^l for $c > \tilde{\alpha}$ is a sub-solution for the null-geodesic equation near the center. Together with the fact that R_h is a super-solution this will proof the theorem. Let be $C(r) := cr^l$ with $c > \tilde{\alpha}$.

Step 1. Expansion of $Y - u$. Evaluating H at $(r, C(r))$ we get

$$H(r, C(r)) = \sum_{i+j\geq 3} C^j T_{ij}\, r^{i+lj} + \sum_{i+j\geq 2} C^{j+1} \Lambda_{ij}\, r^{i+l(1+j)}.$$

For the numbers $i + lj$ and $i + l(1 + j)$, respectively, we have for $\alpha = 0$

$$i + lj \geq l \text{ and } ``=" \Leftrightarrow i = l\ j = 0,$$

$$i + l(1 + j) \geq l + 2 \text{ and } ``=" \Leftrightarrow i = l,\ j = 0.$$

this can be seen by $n(j, k) := (l - 1)j + k$

$$l = n(0, l) < n(l + 1, 0) < \cdots$$

and $m(k, j) := k + l + (l - 1)j$

$$l + 2 = m(2, 0) < m(3, 0) < \cdots < m(l, 0) < m(2, 1) < \cdots.$$

Thus

$$H(r, C(r)) = \tilde{\alpha} r^l + \mathrm{O}(r^{l+1}).$$

It follows:

$$u(r, C) = \mathcal{H}(r, C)^{-1} = \left(\frac{H(r, C)}{cr^l}\right)^{1/2} = \sqrt{\frac{\tilde{\alpha}}{c}} + \mathrm{O}(r^{1/2})$$

and further

$$Y(r, C) - u(r, C) = 1 - \sqrt{\frac{\tilde{\alpha}}{c}} + O(r^{1/2}). \tag{35}$$

Step 2. Evaluation of $\sqrt{\Delta}$. Similarly as above

$$H(r, \tau r^l) = \sum_{i+j\geq 3} \tau^j T_{ij}\, r^{i+lj} + \sum_{i+j\geq 2} \tau^{j+1} \Lambda_{ij}\, r^{i+l(1+j)} = \tilde{\alpha} r^l + O(r^{l+1}).$$

Calculate $(\partial H/\partial r)(r, \tau r^l)$

$$\frac{\partial H}{\partial r}(r, \tau r^l) = \sum_{\substack{i+j\geq 3\\ i\geq 1}} T_{ij} i \tau^j r^{i-1+jl} + \sum_{\substack{i+j\geq 2\\ i\geq 1}} i \Lambda_{ij} \tau^{j+1} r^{i-1+l(1+j)}.$$

For the numbers $n(k, j) := k - 1 + l + (l - 1)j$ and $m(k, j) := k - 1 + (l - 1)j$, we have

$$l - 1 = m(l, 0) < m(3, 1) < m(4, 1) < \cdots$$

and

$$l + 1 = n(2, 0) < n(3, 0) < \cdots.$$

Thus we have

$$\frac{\partial H}{\partial r}(r, \tau r^l) = l\tilde{\alpha} r^{l-1} + O(r^l).$$

We calculate similar to Lemma 4 with $\sigma = \tau r^l$

$$\frac{\partial \mathcal{H}}{\partial r}(r, \sigma)\, d\sigma = -\frac{\sqrt{\sigma}}{2} \frac{(\partial H/\partial r)(r, \sigma)}{H^{3/2}(r, \sigma)}\, d\sigma = -\frac{\sqrt{\tau r^l}}{2} \frac{l\tilde{\alpha} r^{l-1} + O(r^l)}{[\tilde{\alpha}\, r^l + O(r^{l+1})]^{3/2}} r^l\, d\tau.$$

This gives

$$\int_0^{cr^l} \frac{1}{Y(r, \sigma)} \frac{\partial \mathcal{H}}{\partial r}(r, \sigma)\, d\sigma \backsimeq -\int_0^c \frac{\sqrt{\tau r^l}}{2} \frac{l\tilde{\alpha} r^{l-1} + O(r^l)}{[\tilde{\alpha} r^l + O(r^{l+1})]^{3/2}} r^l\, d\tau$$
$$= -\int_0^c \frac{r^{l-1}\sqrt{\tau}}{2} \frac{l\tilde{\alpha} + O(r)}{[\tilde{\alpha} + O(r)]^{3/2}}\, d\tau \quad (\tilde{\alpha} > 0) = -\frac{lr^{l-1}}{2\sqrt{\tilde{\alpha}}} \int_0^c \sqrt{\tau}\, d\tau \backsimeq -\frac{lr^{l-1}}{3} \frac{c^{3/2}}{\sqrt{\tilde{\alpha}}}.$$

Finally with (34)

$$\sqrt{\Delta}(r, \tilde{R}(r)) = \sqrt{\Delta}(r, 0) + \int_0^{cr^3} \frac{1}{Y(r, \sigma)} \frac{\partial \mathcal{H}}{\partial r}(r, \sigma)\, d\sigma$$

$$\geq \xi r^{n-1} - \frac{lr^{l-1}}{3}\sqrt{\frac{c}{\tilde{\alpha}}}c, \quad l \geq 4. \tag{36}$$

Step 3. Conditions for $C(r)$ to be a sub-solution.

According to the definition of a sub-solution for the geodesic equation (14) and using (35) and (36), it is sufficient for $C(r) = cr^l$ to satisfy

$$lcr^{l-1} < \left(1 - \sqrt{\frac{\tilde{\alpha}}{c}}\right)\left(\xi\sqrt{\frac{\tilde{\alpha}}{c}}r^{n-1} - \frac{lr^{l-1}}{3}c\right) \tag{37}$$

for $r > 0$ small.

For $1 \leq n \leq l-1$ there exists $r_n := r(n, l, c, \tilde{\alpha}, \xi) > 0$ such that for every $0 \leq r < r_n$ the inequality is satisfied if $c > \tilde{\alpha}$, and the singularity is naked. Now let $l = n \geq 4$, that is the inequality is independent of r. For convenience define $0 \neq \eta = c/\xi \in \mathbb{R}$, $\varepsilon = \text{sign}(\eta) = \text{sign}(\xi)$ and $\sigma = \sigma(\tilde{\alpha}, c) = \sqrt{\tilde{\alpha}/c} < 1$. By these notations (37) reads as

$$(1-\sigma)\left(\sigma - \frac{l\eta}{3}\right)\begin{cases} < l\eta \text{ if } \xi < 0, \\ > l\eta \text{ if } \xi > 0, \end{cases}$$

or, equivalently,

$$p_\varepsilon(\sigma) = \varepsilon\left(-\sigma^2 + \sigma\left(\frac{l\eta}{3} + 1\right) - l\eta\frac{4}{3}\right) > 0, \tag{38}$$

p_ε has the zeros $\sigma_\pm(\eta) = (1/2)(l\eta/3 + 1) \pm \sqrt{(1/4)(l\eta/3+1)^2 - l\eta(4/3)} \in \mathbb{R}$, whenever $q_l(\eta) := (1/4)(l\eta/3+1)^2 - l\eta(4/3) \geq 0$. The zeros of q_l are positively real

$$\eta_\pm = \frac{21 \pm 12\sqrt{3}}{l} \in \mathbb{R}_+ \quad \forall\, l \geq 4$$

and $q_l < 0$ on $(\eta_-, \eta_+) \subset \mathbb{R}$. Thus $\sigma_\pm \in \mathbb{R}$ if and only if $\eta \in \mathbb{R} \setminus (\eta_-, \eta_+)$. Secondly, observe that $p_+ \leq 0$ everywhere, whenever $\sigma_- = \sigma_+$ ($p_- \geq 0$, respectively), i.e. whenever $q_l = 0$.

Case 1 ($\varepsilon = -1$). It holds $p_- \leq 0$ in $[\sigma_-, \sigma_+]$ and $q_l \neq 0$ for all $\eta \in \mathbb{R}_-$, i.e. $\sigma_- \neq \sigma_+$. Hence, (38) cannot be satisfied there. Therefore, it must hold $(0, 1) \setminus [\sigma_-, \sigma_+] \neq \emptyset$, or equivalently:

$$\sigma_- > 0 \quad \text{or} \quad \sigma_+ < 1. \tag{39}$$

The first condition is clearly equivalent to

$$\frac{1}{4}\left(\frac{l\eta}{3} + 1\right)^2 > \frac{1}{4}\left(\frac{l\eta}{3} + 1\right)^2 - \frac{4l}{3}\eta \geq 0,$$

which is only satisfied if $\eta > 0$. The second condition in (39) reads as

$$\frac{1}{2}\left(\frac{l\eta}{3}-1\right) < -\sqrt{\frac{1}{4}\left(\frac{l\eta}{3}+1\right)^2-\frac{4l}{3}\eta} \le 0$$
$$\Leftrightarrow \frac{1}{4}\left(\frac{l\eta}{3}-1\right)^2 > \frac{1}{4}\left(\frac{l\eta}{3}+1\right)^2-\frac{4l}{3}\eta \Leftrightarrow 0 > -l\eta,$$

whence, we can ignore this case.

Case 2 ($\varepsilon = +1$). With respect to the above discussion let $\sigma_- \neq \sigma_+$, i.e. let $\eta \in \mathbb{R} \setminus [\eta_-, \eta_+]$. Since it holds $p_+ \le 0$ in $\mathbb{R} \setminus (\sigma_-, \sigma_+)$, (38) can never be satisfied there. Hence, it must hold $M := (0, 1) \cap (\sigma_-, \sigma_+) \neq \emptyset$, or equivalently:

$$\sigma_- < 1 \quad \text{and} \quad \sigma_+ > 0. \tag{40}$$

The first condition reads as

$$\frac{1}{2}\left(\frac{l\eta}{3}-1\right) < \sqrt{\frac{1}{4}\left(\frac{l\eta}{3}+1\right)^2-\frac{4l}{3}\eta}.$$

If the left-hand side is non-negative we would obtain after squaring $-l\eta > 0$, and hence it must hold necessarily $\eta < 3/l$. From the second condition in (40) we obtain, arguing similarly as above, that $\eta > -3/l$, and therefore the condition is trivial. We conclude that (38) is satisfied if and only if holds

$$0 < \eta < \eta_- =: \mu \tag{41}$$

(since $\eta_+ < 3/l$ never holds).

The proceeding is now as follows. Let be $\xi, \tilde{\alpha} > 0$ arbitrary. Let be η with (41), that is $M \neq \emptyset$. Further, the function $\sigma(\tilde{\alpha}, \cdot) : (\tilde{\alpha}, \infty) \to (0, 1)$ is one-to-one. By this and by (41) there is some $c > \tilde{\alpha}$, such that there is some $\sigma_0 \in M$ such that holds

$$\sigma_0^2\eta\xi = \sigma_0^2 c = \tilde{\alpha} < \sigma_0^2\mu\xi \Leftrightarrow \frac{1}{\mu} < \frac{1}{\mu\sigma_0^2} < \frac{\xi}{\tilde{\alpha}}.$$

From this inequality we can go backwards, i.e. let be $\xi, \tilde{\alpha}$ with $\xi/\tilde{\alpha} > 1/\mu$. Then there is some $\epsilon > 1$ such that still $\xi/\tilde{\alpha} > \epsilon/\mu$. Choose some $\sigma \in (\max\{\sqrt{1/\epsilon}, \sigma_-\}, 1)$ and set $\eta = \sigma^{-2}\tilde{\alpha}/\xi$. Hence it follows:

$$0 < \eta < \frac{\sigma^{-2}\mu}{\epsilon} < \mu,$$

that is, for η and σ constructed as above, (38) is satisfied, and $c = \sigma^{-2}\tilde{\alpha} > \tilde{\alpha}$ gives rise for some super-solution for the geodesic equation. □

Remark 10. The conditions on ξ and $\tilde{\alpha}$ above include the non-degenerate case, renaming $\tilde{\alpha} = T_{30}$ as in Lemma 2 in [1] and setting $l = 3$. The resulting inequality (37) then is exactly as in the proof there. Observe that $\mu(3) < \xi_c$, for ξ_c the constant given in Theorem 2.

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References

[1] R. Giambo, F. Giannoni, G. Magli, P. Piccione, Commun. Math. Phys. 235 (3) (2003) 545.
[2] R. Giambo, F. Giannoni, G. Magli, P. Piccione, Class. Quant. Gravity 20 (6) (2003) 75.
[3] R. Penrose, Nuovo Cimento 1 (1969) 252.
[4] J. Kijowski, G. Magli, Class. Quant. Grav. 15 (1998) 3891.
[5] G. Magli, Class. Quant. Grav. 14 (1997) 1937.
[6] G. Magli, Class. Quant. Grav. 15 (1998) 3215.
[7] A. Ori, Class. Quant. Grav. 7 (1990) 985.
[8] D.M. Eardley, L. Smarr, Phys. Rev. D 19 (1979) 2239.
[9] D. Christodoulou, Commun. Math. Phys. 93 (1984) 171.